Pre-Calculus 30 Final Exam Review 2020

Chapter 1: Function Transformations

Transformations

- $y = f(x) \rightarrow y = a\,f(b(x - h)) + k$
  - a = vertical stretch by a factor of |a|
  - -a = reflection in the x-axis
  - b = horizontal stretch by a factor of $\frac{1}{|b|}$
  - -b = reflection in the y-axis
  - h = horizontal translation h units right
  - -h = horizontal translation h units left
  - k = vertical translation h units up
  - -k = vertical translation h units down

Mapping

- $(x, y) \rightarrow (\frac{1}{|b|}\,x + h, ay - k)$

Order of transformations

1. Reflections and stretches
2. Translations

Inverse relations

- Exchange the abscissa (x-values) and the ordinate (y-values), f(x) = y → f(y) = x
- Reflected over the diagonal line y = x
- Domain of relation →range of inverse relation
- Range of inverse relation → domain of inverse relation
- Perform vertical line test on a function to determine if it is a function
- Perform horizontal line test on relation to determine if its inverse relation is a function
- If inverse relation is not a function: restrict the domain of the relation to form an inverse function
- Original relations that are functions are represented by f(x) → Inverse functions are represented by $f^{-1}(x)$

Invariant points

- Points that remain unchanged after transformations are applied to it

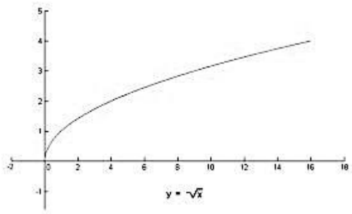
Radical functions

- A function that involves a radical with a variable in the radicand, $y = \sqrt{x}$

Base radical function

❖ $y = \sqrt{f(x)}$

- Left endpoint at (0,0)
- No right endpoint
- Shape is half a parabola
- Domain: {x|x ≥ 0, x∈R}
- Range: {y|y ≥ 0, y∈R}



Transformations on a radical function

❖ $y = \sqrt{x} \rightarrow y = a\sqrt{b(x - h)} + k$

- ‘h’ changes x-intercept and domain
- ‘k’ changes y-intercept and range

Interpreting stretches on a graph of a radical function

- Vertical stretch: substitute an ordered pair into $y = a\sqrt{x}$
- Horizontal stretch: substitute an ordered pair into $y = \sqrt{bx}$

Square root of a function

- $y = \sqrt{f(x)}$ is the square root of $y = f(x)$
- The domain of $y = \sqrt{f(x)}$ consists of all the values where $x \geq 0$
- The range of $y = \sqrt{f(x)}$ consists of the square roots of the values in the range of $y = f(x)$, for which $y = \sqrt{f(x)}$ is defined

Solving radical equations algebraically

- By solving radical equations you are finding the x-intercepts (also known as ‘roots’)
- Extraneous roots are roots discovered algebraically that are not in the domain
- The final answer is written in solution set – only the x-value, not the ordered pair

For questions formatted $a\sqrt{b(x-h)} + k = 0$:

1. Identify any restrictions on the variable in the radicand – set the radicand $(b(x-h)) \geq 0$ and solve
2. Isolate the radical and solve the radical equation, equals to 0 (because @ x-int y = 0), by:
    a) Subtracting k from both sides
    b) Dividing both sides by a
    c) Squaring both sides
    d) Adding h to both sides
3. Determine if the answer from 3. is defined in the answer from 1.
4. If yes, write a solution set with that value, if no, write an empty solution set

For equations formatted $\sqrt{x-h} = x - k$

1. Identify any restrictions on the variable in the radicand – set the radicand $(x-h) \geq 0$ and solve
2. Isolate the radical and solve the radical equation by:
    a) Squaring both sides
    b) Adding/subtracting to get one side to = 0
    c) Factor
3. Determine if the answer from 3. is defined in the answer from 1.
4. Substitute your answers into the original equation to see if you’ve created any extraneous roots
5. Write a solution set

Unit 3: Polynomial Functions

Examples of polynomial functions

- f(x) = 2x – 1
- f(x) = x^2 + 2x - 1
- f(x) = x^3 + x^2 + 2x – 1

Synthetic division

"a" | Coefficients

| Remainder

- Answer is written in the form $polynomial + \frac{remainder}{x-a}$
- Determine restrictions, $x - a \neq 0$ since it is in the denominator

Remainder theorem

- ❖ $P(a) = \frac{P(x)}{x-a}$
- $P(a)$ = remainder
- $P(x)$ = a polynomial

Factor theorem

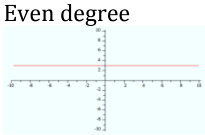
- $x - a$ is a factor of $P(x)$ only when $P(a) = 0$

Factoring higher-degree polynomial - Integral zero theorem states that

- Find factors of the constant and test them to see if you get a remainder of zero

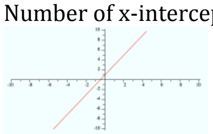
Characteristics of polynomial functions

- Degree 0: Constant Function
  - Number of x-intercepts: 0
  - Even degree



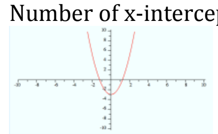
  - ▪ $F(x) = 3$
  - Number of x-intercepts: 0
  - End behaviour: extends from QII to QI
  - Domain: {x|x∈R}
  - Range: {3}

- Degree 1: Linear Function
  - Odd degree
  - Number of x-intercepts: 1



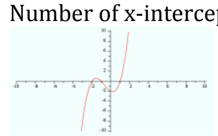
  - ▪ $F(x) = 2x + 1$
  - Number of x-intercepts: 1
  - End behaviour: extends from QIII to QI
  - Domain: {x|x∈R}
  - Range: {y|y∈R}

- Degree 2: Quadratic Function
  - Even degree
  - Number of x-intercepts: 0, 1 or 2



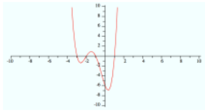
  - ▪ $F(x) = 2x^2 - 3$
  - Number of x-intercepts: 2
  - End behaviour: extends into QI and QII
  - Domain: {x|x∈R}
  - Range: $\{y|y \geq -2, y \in R\}$

- Degree 3: Cubic Function
  - Odd degree
  - Number of x-intercepts: 1, 2 or 3



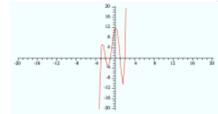
  - ▪ $F(x) = x^3 + 2x^2 - x - 2$
  - Number of x-intercepts: 3
  - End behaviour: QIII and into QI
  - Domain: {x|x∈R}
  - Range: {y|y∈R}

- Degree 4: Quartic Function
  - Even degree
  - Number of x-intercepts: 0, 1, 2, 3 or 4



  - $F(x) = x^4 + 5x^3 + 5x^2 - 5x - 6$
  - Number of x-intercepts: 4
  - End behaviour: QII and ends in QI
  - Domain: $\{x|x \in R\}$
  - Range: $\{y|y \geq -6.91, y \in R\}$

- Degree 5: Quintic Function
  - Odd degree
  - Number of x-intercepts: 1, 2, 3, 4 or 5



  - $F(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12$
  - Number of x-intercepts: 5
  - End behaviour: QIII, ends in QI
  - Domain: $\{x|x \in R\}$
  - Range: $\{y|y \in R\}$

Zeros of a polynomial function

- When a factor is repeated 'n' times, the corresponding zero (root/x-intercept) has a multiplicity of 'n'
- Multiplicity of 1: cross
- Multiplicity is odd $\neq 1$: Tangent and cross
- Multiplicity is even: tangent, no cross

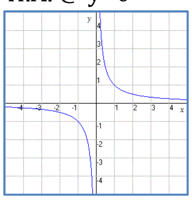
Unit 9: Rational Functions

Rational function

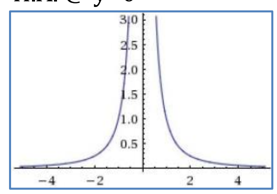
- A function that can be written in the form $f(x) = \frac{p(x)}{q(x)}$, where p(x) and q(x) are polynomials expressions

Base function

❖ $f(x) = \frac{1}{x}$
- $x \neq 0$
- As x → 0, y → $\pm\infty$
- As y → 0, x → $\pm\infty$
- Domain: $\{x | x \neq 0, x \in R\}$
- Range: $\{y | y \neq 0, y \in R\}$
- V.A @ x = 0
- H.A. @ y=0



❖ $f(x) = \frac{1}{x^2}$
- $x \neq 0$
- As x → 0, y → $+\infty$
- As y → 0, x → $\pm\infty$
- Domain: $\{x | x \neq 0, x \in R\}$
- Range: $\{y | y > 0, y \in R\}$
- V.A @ x = 0
- H.A. @ y=0



Asymptotes

- A line that the graph of a relation approaches as a limit or boundary
- Vertical asymptote: x=h, a function never crosses it
- Horizontal asymptote, y=k, a function may or may not cross it

Transformations

❖ $y = \frac{a}{b(x-h)} + k$ or $y = \frac{a}{(b(x-h))^2} + k$
- Translation (h,k): changes position of graph and location asymptotes
- Reflection (-a,-b): changes orientation of graph
- Stretch (a,b): changes shape of graph
- V.A. = h
- V.H = k

Analyzing rational functions

- A vertical asymptote occurs where a non-permissible value exists (in the denominator)
- A point of discontinuity occurs from the common factor that is reduces from the numerator and denominator
- A factor only in the numerator corresponds to an x-intercept
- Both occur where the function is undefined (denominator = 0)
- H.A if the numerator and denominator have the same degree: *y = ratio of coefficients*
- H.A if the degree of the numerator is less than the degree of the denominator: *y=0*

Solving rational equations algebraically

1. Factor the numerator and denominator, note any restrictions
2. Multiply each term in the equation by the LCD in order to reduce the denominator
3. Solve for x
4. Check for extraneous roots
5. Write a solution set

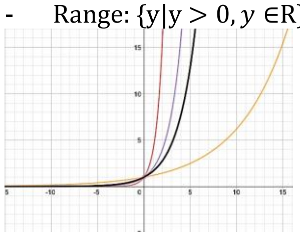
Unit 7: Exponential Functions

Exponential function

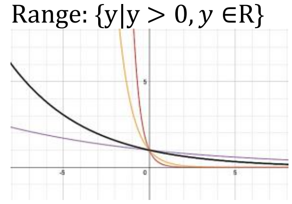
- A function of the form y=c^x, where c is a constant > 0 and x is a variable
- If c = 1: for every value of x, y=1 (horizontal line)

Base function

- y=c^x
- When c>1
  - Increasing from left to right
  - H.A @ y = 0
  - V.A = n/a
  - Larger values of 'c' = steeper curve
  - All pass through (0,1)
  - Point at (1,c)
  - Domain: $\{x|x \in R\}$
  - Range: $\{y|y > 0, y \in R\}$



- When c<1
  - Decreasing from left to right
  - H.A @ y = 0
  - V.A = n/a
  - Smaller values of 'c' = steeper curve
  - All pass through (0,1)
  - Point at (1,c)
  - Domain: $\{x|x \in R\}$
  - Range: $\{y|y > 0, y \in R\}$



Exponent laws

1. $x^a * x^b = x^{a+b}$
2. $(x^a)^b = x^{ab}$
3. $\frac{x^a}{x^b} = x^{a-b}$
4. $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$
5. $(xy)^a = x^a y^a$
6. $x^0 = 1$
7. $x^{\frac{1}{n}} = \sqrt[n]{x}$
8. $x^{\frac{m}{n}} = \sqrt[n]{x^m}\ or\ \left(\sqrt[n]{x}\right)^m$
9. $\left(\frac{x}{y}\right)^{-a} = \left(\frac{y}{x}\right)^a = \frac{y^a}{x^a}$

Exponential growth

- Increasing pattern of values that can be modeled by a function of the form y=c^x, where c>1

Exponential decay

- Decreasing pattern of values that can be modeled by a function of the form y=c^x, where 0<c<1

Half- life

- Length of time for an unstable element to spontaneously decay to one half of its original mass

Transformations

❖ $y = a(c)^{b(x-h)} + k$
- a (affects range, y-int) , b (affects nothing), h (affects y–int), k (affects range, y–int, x-int (if translated down))

Unit 8: Logarithmic function

Logarithmic function

- A function of the form $y = log_c x$, where c>0, ≠1 and x>0

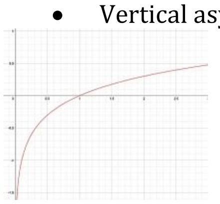
Logarithm

- An exponent
- Inverse is exponential function: $x = c^y$, y is called the logarithm to base c if x

Common logarithm

A logarithm with base 10

- Domain: $\{x|x > 0, x \in R\}$
- Range: $\{y| y \in R\}$
- x-intercept = (1,0)
- Vertical asymptote @ x = 0



Evaluating logarithms

1. Set equal to x
2. Write it in exponential form
3. Use a therefore symbol and exponential laws to determine the value of x
4. Rewrite the original expression and set it equal to the answer found in 3.

Logarithm rules

- $\log_c 1 = 0$, since $c^0 = 0$
- $\log_c c = 1$, since $c^1 = 1$
- $\log_c c^x = x$, since $c^x = c^x$
- $c^{\log_x c} = x, x > 0$, since $\log_c x = \log_c x$

Transformations

- ❖ $y = a \log_c\big(b(x-h)\big) + k$
- a (affects shape), b (affects x-int), h (affects x-int, vertical asymptote, domain), k (affects x-int)

Logarithm laws

- Product law: adding two logs with the same base → $\log M + \log N = \log(MN)$
- Quotient Law: subtracting two logs with the same base → $\log M - \log N = \log(\frac{M}{N})$
- Power Law: multiplying a logarithm by an integer → $P \log M = \log M^P$

Essential Equality statements

- If $\log_c L = \log_c R$, then $L = R$
- $\log_c L = R$, then $L = c^R$
- Only if c, L, R > 0, and c ≠ 0
- Log of 0 or a negative number is undefined

Solving logarithm equations

- If each side contains the same log base, then the expressions are equal
- Change to exponential form and solve
- Isolate logarithm terms on one side of the equation, apply laws of logarithms and solve
- Take the common log of both sides and use laws of logarithms to isolate your variable; then solve with your calculator

Exponential growth a decay questions

1. Take the log of both sides
2. Use laws of logarithms to isolate the variable
3. Solve with calculator
4. Usually, $final\ quantity = initial\ quantity\ (change\ factor)^{number\ of\ changes}$

Unit 4: Trigonometry and the Unit Circle

Angles in standard position

- Vertex is the origin
- Initial side coincides with the positive x-axis
- Angles that rotate counter-clockwise have a positive measure
- Angles that rotate clockwise have a negative measure

Radians

- One radian is the measure of the central angel subtended in a circle by an arc equal in length to the radius of the circle
- $2\pi = 360°$
-

Converting between radians and degrees

- $x°\left(\frac{\pi}{180°}\right)$ (Degrees → radians)
- $x^r\left(\frac{180°}{\pi}\right)$ (Radians → degrees)

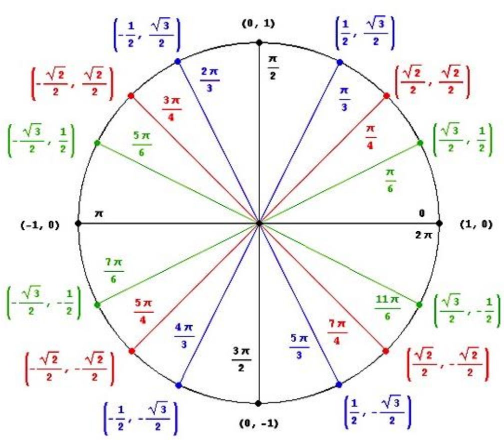
Co-terminal angles

- The infinite number of angles in standard position that have the same initial side
- Created by adding or subtracting multiples of 360° or 2π
- General form: $\theta \pm (360°)n$ or $\theta \pm (2\pi)n, n \in N$

Arc Length of a circle

- $a = \theta r$
  - a = arc lengths
  - θ = Central angle in radians
  - r = Length of the radius

Unit circle



- Equation of Unit Circle: $x^2 + y^2 = 1$ (radius)^2

- $sin\,\frac{\pi}{6} = \frac{1}{2}$
- $cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
- $sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
- $cos\frac{\pi}{3} = \frac{1}{2}$
- $sin\,\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
- $cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

- For any angle in standard position $\theta, r = \sqrt{x^2 + y^2}$
- $(x, y) \rightarrow (cos\theta, sin\theta)$

Primary rations

- $sin\theta = \frac{y}{r}$
- $tan\theta = \frac{y}{x}$
- $cos\theta = \frac{x}{r}$
- SYR CXR TYX

Reciprocal ratios

- $sin\theta = \frac{1}{csc\theta}$
- $cos\theta = \frac{1}{sec\theta}$
- $tan\theta = \frac{1}{cot\theta}$

CAST (Cos, All, Sin, Tan)

- 1st quadrant = ALL trig functions have positive values
- 2nd quadrant = SINE function has positive values
- 3rd quadrant = TANGENT function has positive values
- 4th quadrant = COSINE function has positive values

S	A
T	C

Reference Angles

- Always positive

Finding angles, given their trigonometric ratios

1. Use calculator or unit circle to find the value of the angle
2. Use CAST to determine all the angles that have the angle from 1. as a reference angle, be sure the angles satisfy the domain

Interval notation

- $[0,2\pi) = 0 \leq \theta < 2\pi$

Trigonometric equation

- Equation involving trigonometric ratios

1. Isolate the trigonometric ratio, for quadratic trig ratios you will need to factor or use the square root property
2. Consider what angles produce the value from 1., while satisfying the domain

Unit 5: Trigonometric Functions and Graphs

Periodic function

- A function that repeats over regular intervals (cycles) of its domain

Period

- Horizontal length of one cycle

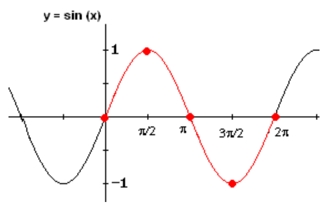
Sinusoidal curve

- A curve that oscillates repeatedly up and down from a center line

Amplitude

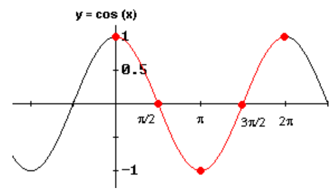
- The maximum vertical distance the graph of a sinusoidal function varied above and below the horizontal central axis of the curve

Graph of the sine function



- Periodic and continuous
- Starts at midpoint, rises to maximum value, returns to midpoint, drops to minimum value, returns to midpoint
- Domain: $\{x|x \in R\}$
- Range: $\{y|-1 \leq y \leq 1, y \in R\}$
- Maximum value: 1
- Minimum value: -1
- Amplitude: 1
- Period length: 2π
- y-intercept: (0,0)

Graph of the cosine function



- Periodic and continuous
- Start at maximum value, drops to midpoint, drops to minimum value, rises to mid point, rises to maximum value
- Domain: $\{x|x \in R\}$
- Range: $\{y|-1 \leq y \leq 1, y \in R\}$
- Maximum value: 1
- Minimum value: -1
- Amplitude: 1
- Period length: 2π
- y-intercept: (0,1)

Transformations

- ❖ $y = a\sin b(x - c) + d$ or $y = a\cos b(x - c) + d$
- 'a' = vertical stretch, corresponds to amplitude
  - $A = \frac{Maximum\ value - minimum\ value}{2}$
  - Always positive
  - Affects range
  - If a<0, reflection in x-axis
- 'b' = horizontal stretch, influences period
  - $per = \frac{2\pi}{|b|}$
  - If b<0, reflection in y-axis
- 'c' = phase shift
  - Translated right if c>0
  - Translated left if c<0

- ‘d’ = vertical translation (midline)
  - $d = \frac{M_{aximum\ value} + m_{inimum\ value}}{2}$
  - Translated up if d>0
  - Translated down if d<0
- Maximum value
  - $M = d + |a|$
- Minimum value
  - $m = d - |a|$

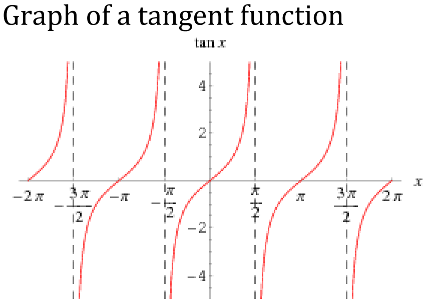
Graphing sin and cos functions

1. Determine: Amplitude, Period, Midline (d), Phase shift (c), Max and Min values
2. Divide the period value into four segments, label
3. Plot points determined in 1.
4. Connect the points with a curved line

Determining the equation from a graph

1. Find the minimum and maximum value, then use to determine ‘a’ and ‘d’
2. Determine the period length, then use to determine ‘b’
3. For a sin function: locate the nearest midpoint of a cycle and determine the displaced distance from y=0. For a cos function: locate the first maximum point of a cycle and determine the displaced distance from y=0.

Graph of a tangent function



- D: $\{x|x \neq \frac{\pi}{2} + n\pi, n \in I, x \in R\}$
- R: $\{y|y \in R\}$
- Maximum value: n/a
- Minimum value: n/a
- Amplitude: 1
- Period length: π
- y-intercept: (0,0)
- x- intercept: $x = n\pi, n \in I$
- V.A @ $x = \frac{\pi}{2} + n\pi, n \in I$

Unit 6: Trigonometric Identities

The Reciprocal Identities

- $csc\theta = \frac{1}{sin\theta}$
- $sec\theta = \frac{1}{cos\theta}$
- $cot\theta = \frac{1}{tan\theta}$

The Quotient Identities

- $tan\theta = \frac{sin\theta}{cos\theta}$
- $cot\theta = \frac{cos\theta}{sin\theta}$

The Pythagorean Identities

- $\cos^2 x + \sin^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $\cot^2 x + 1 = \csc^2 x$

Proving expressions with identities

1. Simplify one side of equation only
2. Use the fundamental identities as proof
3. Once both sides are equal, write LHS=RHD