Chapter 4 Quadratic Equations

Section 4.1 Solving Quadratic Equations by Graphing

Definitions

Quadratic equation:

Roots of an equation:

Zeros of a function:

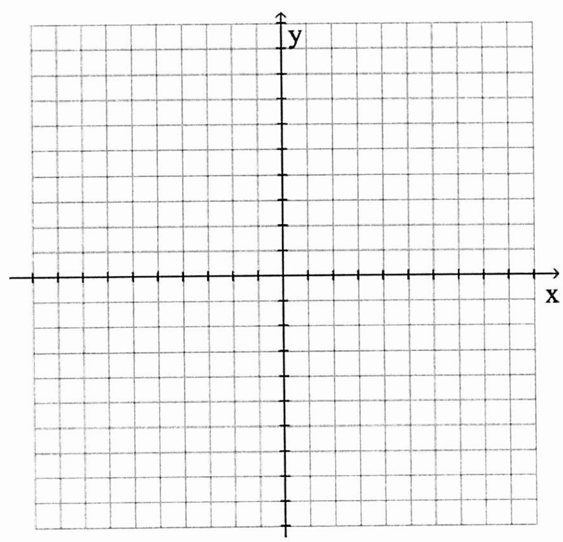
Example1:

Given the quadratic equation $0 = x^2 + 3x + 2$, what are the roots of the equation?

Method 1: Graphing by Table of Values
One way is to graph the function $f(x) = x^2 + 3x + 2$ and determine the x-intercepts.

Table of Values Method

x	y



Solution: { , }

Example2:

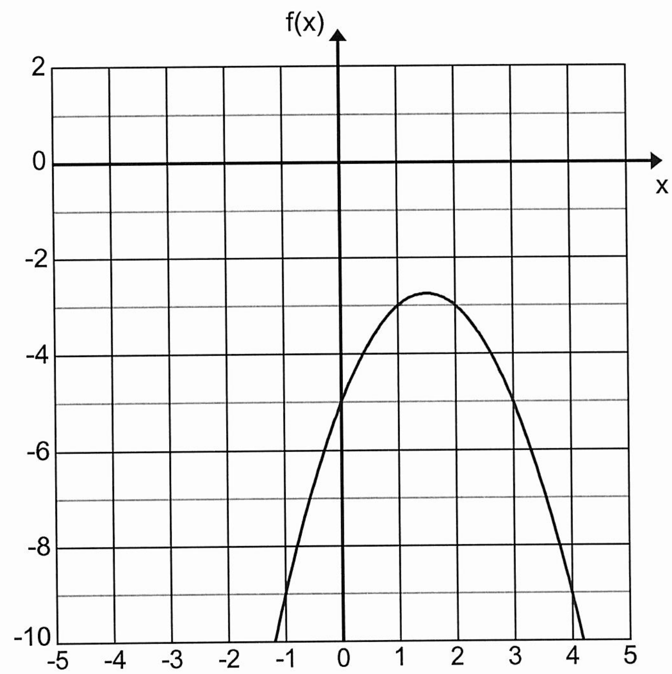
Given the quadratic equation $0 = -x^2 + 3x - 5$, what are the roots of the equation?

Method 2: Graphing with technology.
One way is to graph the function $f(x) = -x^2 + 3x - 5$ using technology. Computer programs make it easy to graph equations very quickly.

If we look at the graph after it is drawn using technology, it can be very quick to spot the roots.

In this case, since there are no x-intercepts at all, there are no roots.

That is, there are no values for x that will make the function $f(x) = 0$.



What are the possible situations for the number of roots?

zero roots = no x-intercepts

one root = one x-intercept

two roots = two x-intercepts

Desmos Graphing Calculator App (iPhone or Android) **Free!**

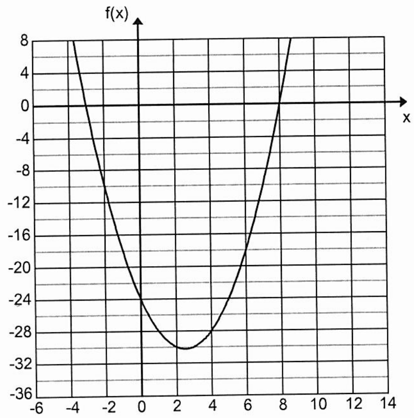
www.desmos.com/calculator (on the computer)

Textbook Assignment: Page 215-216 #1, 2, 3, 4, 5, 8, 9, 10

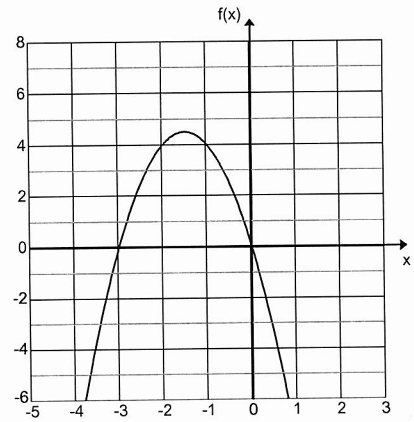
Section 4.1 Pages 215-216

Graphs for Questions on Pages 215-216 in Textbook

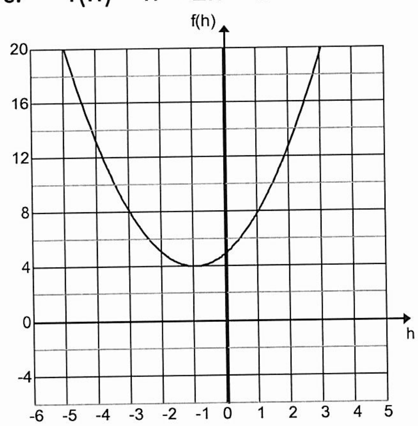
#3a. $f(x) = x^2 - 5x - 24$



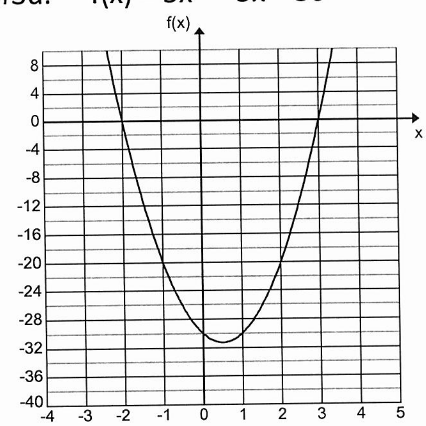
#3b. $f(r) = -2r^2 - 6r$



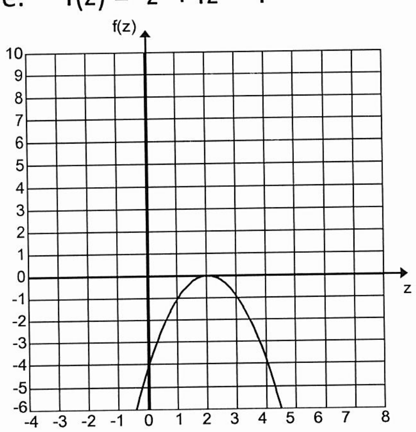
#3c. $f(h) = h^2 + 2h + 5$



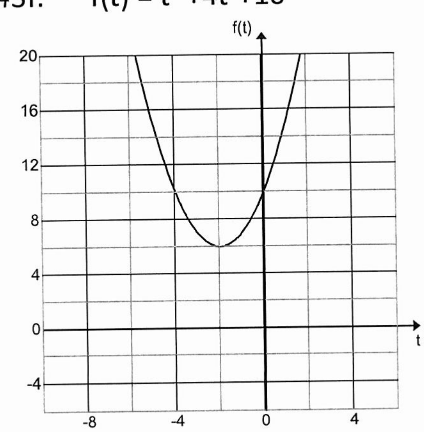
#3d. $f(x) = 5x^2 - 5x - 30$



#3e. $f(z) = -z^2 + 4z - 4$

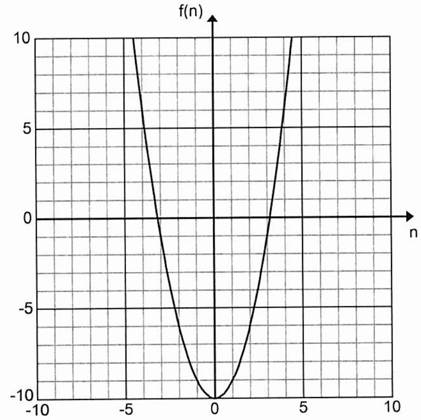


#3f. $f(t) = t^2 + 4t + 10$

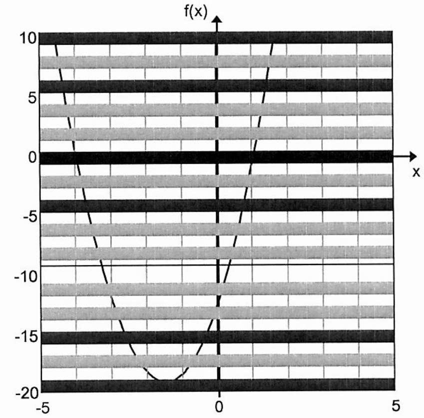


Section 4.1 Pages 215-216
Graphs for Questions on Pages 215-216 in Textbook

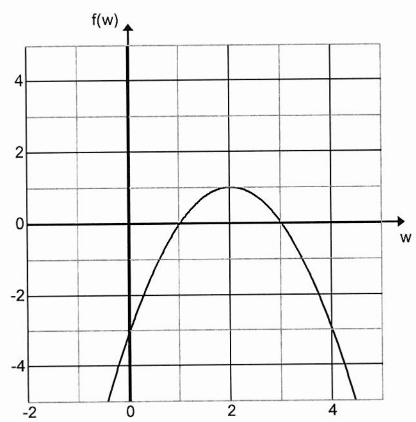
#4a. $f(n) = n^2 - 10$



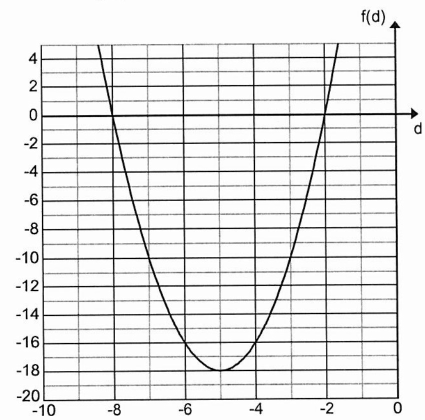
#4b. $f(x) = 3x^2 + 9x - 12$



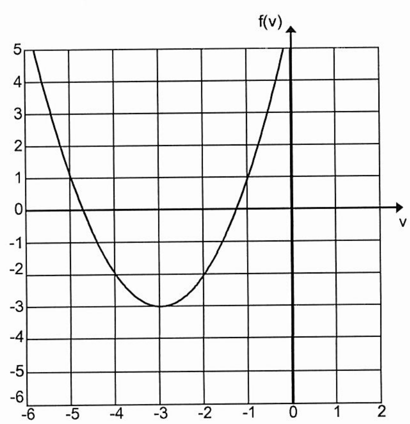
#4c. $f(w) = -w^2 + 4w - 3$



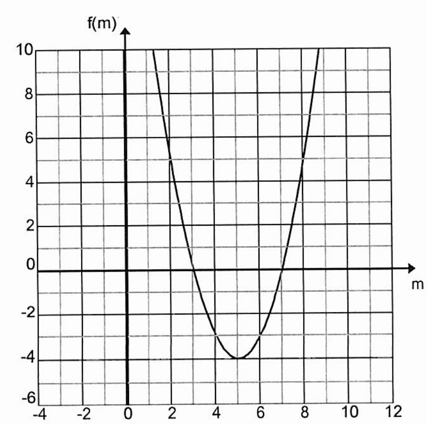
#4d. $f(d) = 2d^2 + 20d + 32$



#4e. $f(v) = v^2 + 6v + 6$

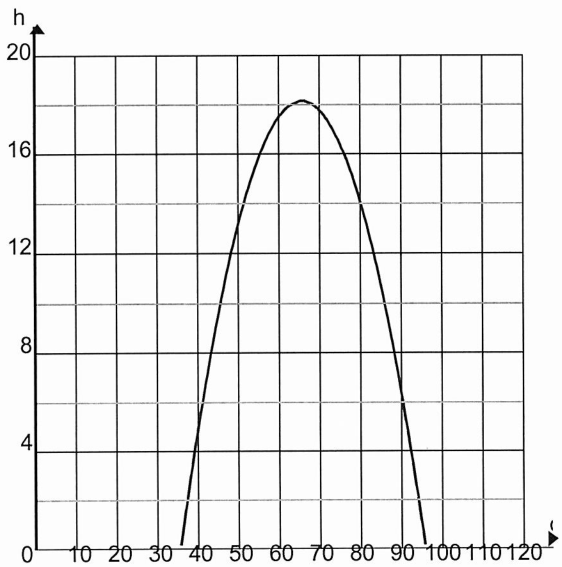


#4f. $f(m) = m^2 - 10m + 21$

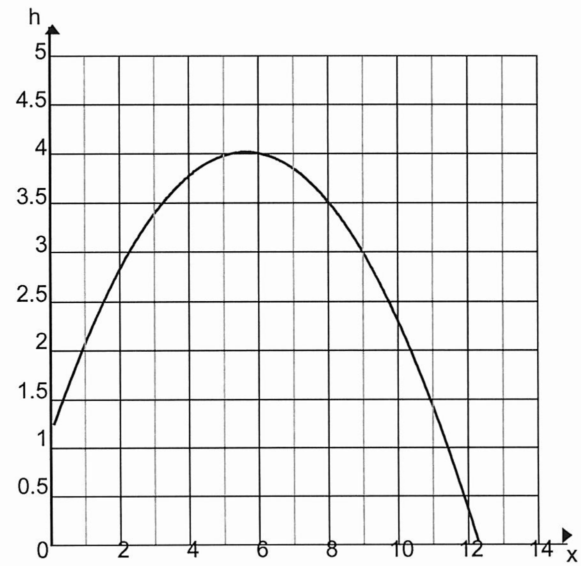


Section 4.1 Pages 215-216
Graphs for Questions on Pages 215-216 in Textbook

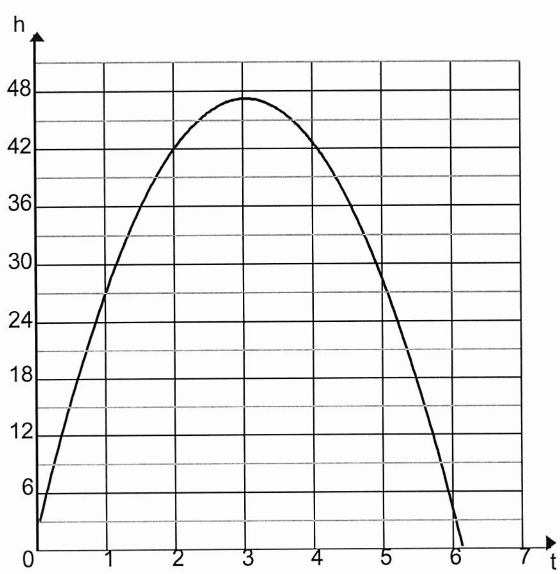
#5. $h(d) = -0.02d^2 + 2.6d - 66.5$



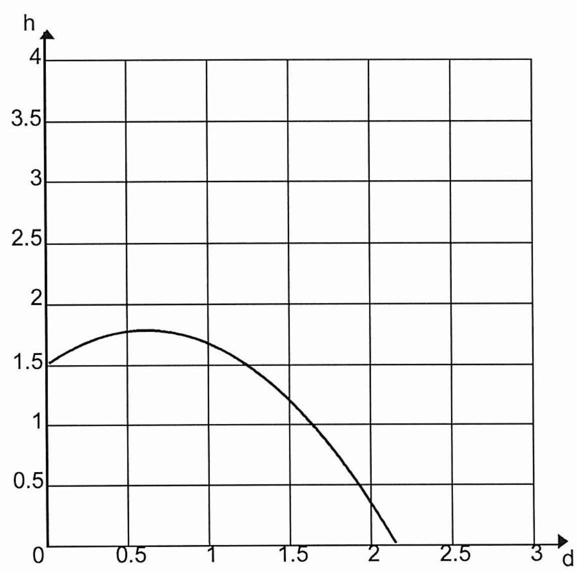
#8. $h(x) = -0.09x^2 + x + 1.2$



#9. $h(t) = -4.9(t - 3)^2 + 47$



#10. $h(d) = -0.75d^2 + 0.9d + 1.5$



Math PreCalc 20 Chapter 4 Review of Factoring

Multiplying (Expanding)	**Factoring (Dividing)**
Type 1: Monomial x Binomial **Monomial x Trinomial** Ex: $3(x + 4) = 3x + 12$ $-2(x^2 + 2x - 1) = -2x^2 - 4x + 2$ <u>Questions to try</u> Multiply the following: 1. $5(x - 3)$ 2. $x(x + 5)$ 3. $2x(x^2 - 3x + 2)$ 4. $-4(x - 5)$ 5. $-3x(x + 2y)$	**Type 1: Factoring GCF** Ex: $2x + 10 = 2(x + 5)$ Note: GCF is 2 $-3x^2 + 6x = -3x(x - 2)$ Note: GCF is $-3x$ <u>Questions to try</u> Factor the following: 1. $4x + 12$ 2. $x^2 - 6xy$ 3. $10x^2 + 20x$ 4. $-5x - 10$ 5. $-4x^2y + 8xy$
Type 2: Binomial x Binomial **Pattern is $(a + b)(a - b) = a^2 - b^2$** Ex: $(x + 7)(x - 7) = x^2 - 49$ $(2y + 3)(2y - 3) = 4y^2 - 9$ <u>Questions to try</u> Multiply the following: 1. $(x + 6)(x - 6)$ 2. $(3y + 5)(3y - 5)$ 3. $(2 + x)(2 - x)$ 4. $(2x + 5y)(2x - 5y)$ 5. $(x + 2y)(x - 2y)$	**Type 2: Factoring Difference of 2 Squares** **Pattern is $a^2 - b^2 = (a + b)(a - b)$** Ex: $x^2 - 4 = (x + 2)(x - 2)$ $25y^2 - 16z^2 = (5y + 4z)(5y - 4z)$ <u>Questions to try</u> Factor the following: 1. $x^2 - 25$ 2. $100x^2 - 9$ 3. $49 - y^2$ 4. $16x^2 - 81z^2$ 5. $y^2 - 9z^2$

Type 3: Binomial x Binomial
$(x \pm ?)(x \pm ?)$

Pattern: Multiply 2 numbers to get the last term
Add 2 numbers to get the middle term

Or

Use FOIL (Double Distributive Property)

Ex: $(x + 3)(x + 4) = x^2 + 7x + 12$
$(x - 3)(x - 4) = x^2 - 7x + 12$
$(x + 3)(x - 4) = x^2 - 1x - 12$ or $x^2 - x - 12$
$(x - 3)(x + 4) = x^2 + 1x - 12$ or $x^2 + x - 12$

<u>Questions to try</u>
Multiply the following:

1. $(x + 2)(x + 5)$
2. $(y + 1)(y + 7)$
3. $(x - 5)(x - 7)$
4. $(y - 2)(y - 3)$
5. $(x + 2)(x - 4)$
6. $(x + 6)(x - 8)$
7. $(y - 5)(y + 1)$
8. $(y - 8)(y + 10)$

Type 3: Factoring Trinomials
($1x^2$ only…$x^2 \pm ?x \pm ?$)

Pattern: Find 2 numbers whose product is last #
Same 2 numbers add up to middle #

Ex: $x^2 + 8x + 15 = (x + 5)(x + 3)$
$x^2 - 8x + 15 = (x - 5)(x - 3)$
$x^2 - 2x - 15 = (x - 5)(x + 3)$
$x^2 + 2x - 15 = (x + 5)(x - 3)$

<u>Questions to try</u>
Factor the following:

1. $x^2 + 10x + 16$
2. $y^2 + 9y + 18$
3. $x^2 - 7x + 10$
4. $y^2 - 10y + 9$
5. $x^2 - 3x - 28$
6. $x^2 - 7x - 18$
7. $y^2 + 2y - 63$
8. $y^2 + 5y - 36$

Type 4: Combos of Types 1,2,3 (GCF + one other)

Ex: $2x^2 - 8 = 2(x^2 - 4)$
$= 2(x + 2)(x - 2)$

$x^3 + 3x^2 - 10x = x(x^2 + 3x - 10)$
$= x(x + 5)(x - 2)$

Type 4: Combos of Types 1,2,3 (GCF + one other)

<u>Questions to try</u> Factor the following:

1. $3x^2 - 27$
2. $10x^2 + 20x - 150$

Type 5: Binomial x Binomial
(?x ± ?)(?x ± ?)

Use FOIL (double distributive property)
or
Box Method

Not both 1x's here anymore!

Ex: $(2x + 3)(3x + 4)$
FOIL: $6x^2 + 8x + 9x + 12$ (**F**irst,**O**utside,**I**nside,**L**ast)
$6x^2 + 17x + 12$

OR (Box Method)

	$2x$	$+3$
$3x$		
$+4$		

	$2x$	$+3$
$3x$	$\mathbf{6x^2}$	$\mathbf{9x}$
$+4$	$\mathbf{8x}$	$\mathbf{12}$

$= 6x^2 + 9x + 8x + 12$
$= 6x^2 + 17x + 12$

<u>Questions to try</u>
Multiply the following:

1. $(2x + 5)(x + 3)$
2. $(3x + 2)(2x + 1)$
3. $(x - 4)(3x - 5)$
4. $(2x + 1)(3x - 2)$

Type 5: Factoring Trinomials
($?x^2 \pm ?x \pm ?$)

Use Trial_and_Error Method
or
Box Method

Not just $1x^2$ here anymore!

Ex: $2x^2 + 5x + 2 = (2x + 1)(x + 2)$
(Try numbers and Test by Multiplying until it works.)
(This method is only good if numbers are small!)

OR (Box Method)

$2x^2 + 5x + 2$
$4x^2$
$4x \quad 1x$

Multiply the two end terms.

Now find two numbers that multiply to get $4x^2$ and add to get $5x$.
Fill Box with these values.

$2x^2$	$4x$
$1x$	2

Now find GCF of 2 rows and 2 columns.

	x	**$+2$**
$2x$	$2x^2$	$4x$
$+1$	$1x$	2

<u>Questions to try</u>
Factor the following:

1. $2x^2 + 13x + 15$
2. $6x^2 + 11x + 3$
3. $2x^2 - 3x - 9$
4. $3x^2 - 19x + 20$

Type 5 continued: Factoring Trinomials ($a \neq 1$) (**Guess Method**)

- Remember that the general form of a trinomial is $ax^2 + bx + c$

- **Examples of FOIL with this type**

  $(3x + 2)(2x + 1) = 6x^2 + \mathbf{3x + 4x} + 2 = 6x^2 + \mathbf{7x} + 2$
  $(3x - 2)(2x - 1) = 6x^2 \mathbf{- 3x - 4x} + 2 = 6x^2 \mathbf{- 7x} + 2$

  $(5x + 7)(2x - 3) = 10x^2 \mathbf{- 15x + 14x} - 21 = 10x^2 \mathbf{- x} - 21$
  $(5x - 7)(2x + 3) = 10x^2 \mathbf{+ 15x - 14x} - 21 = 10x^2 \mathbf{+ x} - 21$

- Remember the following patterns from Section 2.5:

  $x^2 + \square x + \square = (x + \_\_)(x + \_\_)$
  $x^2 - \square x + \square = (x - \_\_)(x - \_\_)$

  Note: If the **last term** is **pos, signs** are the **same**!

  $x^2 + \square x - \square = (x + \_\_)(x - \_\_)$
  $x^2 - \square x - \square = (x - \_\_)(x + \_\_)$

  Note: If the **last term** is **neg, signs** are **opposite**!

- **Factoring Trinomials Using Intelligent Guessing Method**
  (We use this method with the 1st and last terms have very few factors, especially if the numbers are prime!)

  $2x^2 - 7x + 5 = (\_\_x - \_\_)(\_\_x - \_\_)$ Write **pattern** of signs
  $= (2x - \_\_)(x - \_\_)$ Choose **factors** of **1st** term.
  $= (2x - 1)(x - 5)$ Choose **factors** of **last** term.

  Quickly Foil to check this guess. If it doesn't work, try another guess by **rearranging the factors** of the last term. In this case the middle term will become **-11x** which is not correct!

  $= (2x - 5)(x - 1)$

  Quickly Foil again to check. If it works, then you are done. In this case the middle term will become **-7x** which is correct!

Factor the following using the Guess Method:

$2x^2 + 5x + 2 = (\_\_x + \_\_)(\_\_x + \_\_)$ Notice that the signs have to be positive!

$3x^2 + 4x - 7 = (\_\_x + \_\_)(\_\_x - \_\_)$
$= (3x + \_\_)(x - \_\_)$
OR $= (3x - \_\_)(x + \_\_)$

Notice how there are 2 choices for the signs!
Only ONE will be correct!!

$3x^2 - x - 2 = (\_\_x + \_\_)(\_\_x - \_\_)$

Type 5 continued: Factoring Trinomials ($a \neq 1$) (**Box Method**)

Ex1: Factor $3x^2 + 16x + 20$

$3x^2$	___x
___x	20

Step 1. Fill in the 1st and last terms into the boxes shown. Notice that the other 2 boxes will both have "x" terms.

Note: $3 \cdot 20 = 60$ We need this for step 2!

$3x^2$	$10x$
$6x$	20

Step 2. Fill in the final 2 boxes with 2 numbers that multiply to 60. Remember that when you Cross Multiply, each must be the same result.
These numbers must also add up to 16. (Middle Term)
$6 \cdot 10 = 60$ $\quad$ $6 + 10 = 16$

	$3x$	$+10$
x	$3x^2$	$10x$
$+2$	$6x$	20

Step 3. Now factor out the GCF of each row and each column to product the final answer.

The final answer is $3x^2 + 16x + 20 =$ $\mathbf{(3x + 10)(x + 2)}$

Check by using FOIL to confirm your answer!

Try the following 3 examples (Note: All signs are positive for these 3 problems.)

#1. $15x^2 + 16x + 4$ $\quad$ #2. $6x^2 + 11x + 4$ $\quad$ #3. $6x^2 + 11x + 5$

We can use the Box Method to Factor any of the more difficult types (even those with negative values):
{**KEY NOTE**: There cannot be any GCF factors in the original question for this method to work!!}

Ex2: Factor $2x^2 - x - 15$

$2x^2$	___x
___x	-15

Step 1. Fill in the 1st and last terms into the boxes shown. Notice that the other 2 boxes will both have "x" terms.

Note: $2 \cdot -15 = -30$ We need this for step 2!

$2x^2$	-6x
5x	-15

Step 2. Fill in the final 2 boxes with 2 numbers that multiply to -30. Remember that when you Cross Multiply, each must be the same result.
These numbers must also add up to -1. (Middle Term)
$5 \cdot -6 = -30$ $\quad$ $5 + -6 = -1$

	x	-3
2x	$2x^2$	-6x
+5	5x	-15

Step 3. Now factor out the GCF of each row and each column to product the final answer. If the first number is negative in that row or column, factor out the negative also!

The final answer is $2x^2 - x - 15 =$ **(x – 3)(2x + 5)**

Check by using FOIL to confirm your answer!

<u>**Practice**</u>
Factor the following: (Watch for GCF first, then Diff of 2 Squares, then Trinomials!!)

1. $x^2 - 4$
2. $x^2 - x - 6$
3. $6x^2 + 7x - 2$
4. $2x^2 - 6x - 56$
5. $8x^2 + 2x - 3$
6. $3x^2 - 6x$
7. $2x^2 - 13x + 15$
8. $6x^2 - 13x + 6$

Math PreCalc 20 Extra Practice Factoring
(Answers on the next page!)

Factor the following. (Watch for all the different types!)

1. $5x^2 + 17x + 12$

2. $3x^2 - 16x + 21$

3. $x^2 - 36$

4. $10x^2 - 17x + 3$

5. $3x^2 + 19x + 30$

6. $2x^2 - 5x - 25$

7. $9x^2 - 4$

8. $3x^2 + 8x - 11$

9. $x^2 + 6x + 8$

10. $2x^2 - 15x + 28$

Solutions to previous questions:

1. $5x^2 + 17x + 12$
   $(5x + 12)(x + 1)$

2. $3x^2 - 16x + 21$
   $(3x - 7)(x - 3)$

3. $x^2 - 36$
   $(x - 6)(x + 6)$

4. $10x^2 - 17x + 3$
   $(5x - 1)(2x - 3)$

5. $3x^2 + 19x + 30$
   $(x + 3)(3x + 10)$

6. $2x^2 - 5x - 25$
   $(2x + 5)(x - 5)$

7. $9x^2 - 4$
   $(3x + 2)(3x - 2)$

8. $3x^2 + 8x - 11$
   $(x - 1)(3x + 11)$

9. $x^2 + 6x + 8$
   $(x + 4)(x + 2)$

10. $2x^2 - 15x + 28$
    $(x - 4)(2x - 7)$

Box Method Solutions for some of the above:

#1. $5x^2 + 17x + 12$

	x	+1
5x	$5x^2$	5x
+12	12x	12

#2. $3x^2 – 16x + 21$

	3x	-7
x	$3x^2$	-7x
-3	-9x	21

#4. $10x^2 – 17x – 3$

	5x	-1
2x	$10x^2$	-2x
-3	-15x	-3

#6. $2x^2 – 5x – 25$

	2x	+5
x	$2x^2$	5x
-5	-10x	-25

Section 4.2 Solving Quadratic Equations by Factoring

Once we are skilled with the basic types of factoring (GCF, Diff of 2 Squares, and Trinomials), we can deal with more advanced types of questions.

Using Fractional GCF's with Trinomials:

$\frac{1}{4}x^2 - x - 3$ $\frac{1}{2}x^2 - x - 4$ $x^2 - \frac{5}{2}x - 6$

Fractional or Decimal Difference of 2 Squares:

$\frac{1}{4}x^2 - 9y^2$ $0.25x^2 - 1.44$

Advanced Trinomials and Difference of Squares

$(x-1)^2 + 4(x-1) + 3$ $12(x+2)^2 + 24(x+2) + 9$

$(2x+1)^2 - (4x-3)^2$

Textbook Assignment: Page 229-230 #1ab, 2ab, 3all, 4all, 5all, 6ab

Section 4.2 continued Solving Quadratic Equations by Factoring

Zero Property:

Solve the following Basic Types of Quadratic Equations by Factoring:

$(x + 2)(x - 3) = 0$

$(3x + 1)(2x - 5) = 0$

$x^2 + 2x = 8$

$x^2 - 100 = 0$

$9y^2 - 144 = 0$

$2x^2 - 5x + 2 = 0$

Write the quadratic equation that has the roots given:

Roots are -4 and 5

Roots are $\frac{2}{3}$ *and* $-\frac{4}{5}$

Solve the following More Advanced Types of Quadratic Equations by Factoring:

$-2(x+3)^2 + 12(x+3) + 14 = 0$

A waterslide ends with a slider dropping into a deep pool. The path of the slider after leaving the slide can be approximated by the quadratic function below:

$$h(d) = -\frac{1}{6}d^2 - \frac{1}{6}d + 2$$

where *h* is the height above the surface of the pool and *d* is the horizontal distance the slider travels after leaving the slide, both in feet.

What is the horizontal distance the slider travels before hitting the water?

Textbook Assignment: Page 230-232 #7all, 10abc, 9abcd, 8all, 11, 13, 17, 20

Section 4.3 Solving Quadratic Equations by Completing the Square

We can solve a quadratic equation that is in the form $ax^2 \pm c = 0$ (notice that there is no "x-term") by using a technique called the Square Root Method (square root both sides). This method can be used when factoring is not possible. (Note: Factoring is always the fastest method to use if possible!!) Solve the following quadratic equations:

$2x^2 - 20 = 0$ $\qquad$ $-3x^2 + 18 = 0$ $\qquad$ $(x + 3)^2 = 16$

We cannot use the above method if there is an "x-term" that is not squared.
For example, $x^2 + x = 3$ {We cannot use the square root method for this!}

However, always try to use factoring since **factoring is much quicker and simpler**.

Solving by Completing the Square

Another method used to solve Quadratic Equations is called "Completing the Square". Before using this method, we must recall an important factoring skill:

Factor these trinomials:

$x^2 + 10x + 25$	$x^2 + 20x + 100$	$x^2 - 6x + 9$	$2x^2 + 8x + 8$
$(x + 5)(x + 5)$	$(x + 10)(x + 10)$	$(x - 3)(x - 3)$	$2(x^2 + 4x + 4)$
$(x + 5)^2$	$(x + 10)^2$	$(x - 3)^2$	$2(x + 2)(x + 2)$
			$2(x + 2)^2$

Do you see a pattern between the value in front of the "x-term" and the "final value (constant)"?
$x^2 + bx + c$ $\quad$ The "c" value is half the "b" value squared. $(b \div 2)^2 = c$

Below is an example that can be done by factoring, but we will use it to introduce how to complete the square for quadratic equations of the type $ax^2 + bx + c$ where $a = 1$ (simpler type).

Ex1: $x^2 - 24 = -10x$

$x^2 + 10x \qquad = 24$	Step1: Put the c value to one side (opposite x^2 and x terms) (Leave room to complete the square on the left side)
$x^2 + 10x + 25 = 24 + 25$	Step2: Complete the square $c=(b \div 2)^2$ or (half of b)^2 Add c to both sides
$(x + 5)^2 = 49$	Step3: Factor the completed square Simplify right side
$x + 5 = \pm\sqrt{49}$	Step4: Square root both sides (Consider both the positive and negative square roots)
$x + 5 = \pm 7$	
$x + 5 = 7$ OR $x + 5 = -7$	Step5: Solve the 2 equations
$x = 2$ OR $x = -12$	Step6: Mental Check to verify answers

Find the solutions in EXACT FORM. (Rounding answers is not acceptable!)

$w^2 - 4w - 11 = 0$ (note: We can't factor this question so we must use a different method)
Can we use the Square Root method immediately?

$x^2 + 5x - 7 = 0$

What if in Ex3 it was +7 instead?

Practice Questions

(If you can't immediately use the **Square Root Method**, you must **Complete the Square** first!)

1. $2x^2 = 64$
2. $2x^2 - 6 = 0$
3. $(x - 3)^2 = 16$
4. $(x + 2)^2 = 10$

5. $x^2 + 6x + 2 = 0$
6. $y^2 - 10y - 15 = 0$
7. $x^2 + 2 = -3x$
8. $x^2 - 5x + 1 = 0$

Section 4.3 Solving Quadratic Equations by Completing the Square

$(ax^2 + bx + c)\ when\ a \neq 1$

$2x^2 + 8x + 1 = 0$

$3x^2 + 9x + 2 = 0$

Textbook Assignment: Page 240-241 #1ab, 4ab, 5abc, 6abcd, 7ab

Section 4.4 Solving Quadratic Equations using the Quadratic Formula

Quadratic equations can be solved using the quadratic formula.
If the quadratic equation is in standard form $ax^2 + bx + c = 0$, the quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Discriminant

The discriminant is ____.

The discriminant can be used to ____.

Case1: If the discriminant is positive: ______________________________

Case2: If the discriminant is zero: __________________________________

Case3: If the discriminant is negative ______________________________

Determine the nature of the roots (number of solutions) for the following:

$3x^2 + 5x - 2 = 0$ $\quad\quad$ $x^2 + 4x + 4 = 0$ $\quad\quad$ $2x^2 + 3x + 4 = 0$

Solve the following using the quadratic formula. Leave answers in EXACT FORM.

$\frac{1}{4}x^2 - 3x + 9$ $\quad\quad$ $x^2 = 2x + 1$

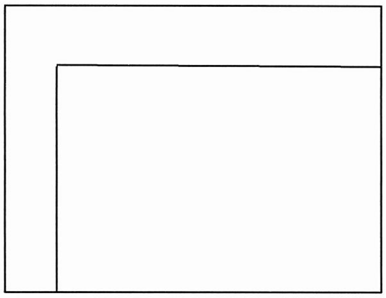
Textbook Assignment: Page 254 1ade, 2ace, 3abcd, 4abc, 5ab, 6

Word Problem:

A picture measures 30cm by 21cm. You crop the picture by removing strips of the same width from the top and one side of the picture. This reduces the area to 40% of the original area. Determine the width of the removed strips.

Let x be the width of the removed strips.

Draw the diagram.



Textbook Assignment: Page 254-255 #8, 9, 11, 13

Example2:

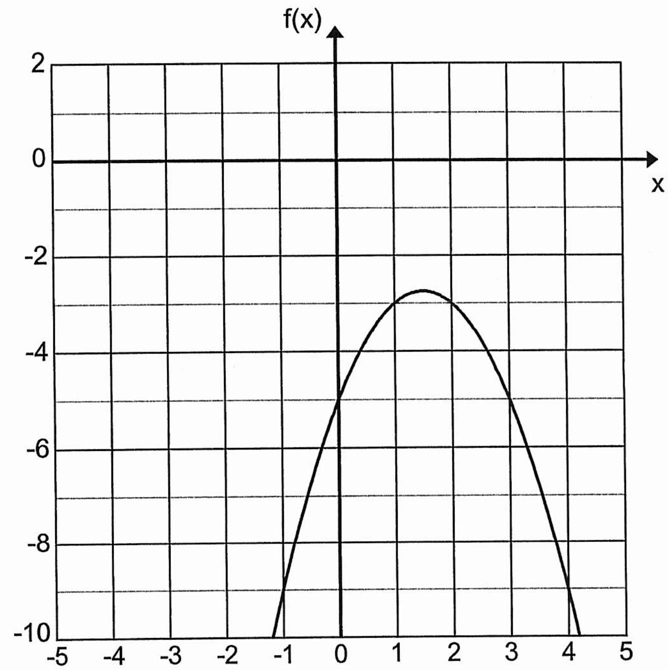
Given the quadratic equation $0 = -x^2 + 3x - 5$, what are the roots of the equation?

Method 2: Graphing with technology.
One way is to graph the function $f(x) = -x^2 + 3x - 5$ using technology. Computer programs make it easy to graph equations very quickly.

If we look at the graph after it is drawn using technology, it can be very quick to spot the roots.

In this case, since there are no x-intercepts at all, there are no roots.

That is, there are no values for x that will make the function $f(x) = 0$.



What are the possible situations for the number of roots?

zero roots = no x-intercepts

one root = one x-intercept

two roots = two x-intercepts

Desmos Graphing Calculator App (iPhone or Android) **Free!**

www.desmos.com/calculator (on the computer)

Textbook Assignment: Page 215-216 #1, 2, 3, 4, 5, 8, 9, 10